EQUIVALENCE BETWEEN THE EXTENDED WINDOW DESIGN OF IIR FILTERS AND LEAST SQUARES FREQUENCY DOMAIN DESIGNS

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ABSTRACT

A new look at the problem of digitizing analog filters is taken in the light of the proof of the equivalence between weighted least squares (WLS) frequency-domain methods and a previously proposed interpolation-based method referred to as the extended window design (EWD). This equivalence is used to prove new properties of the EWD and WLS methods. First, the EWD filters are shown to be near-optimal due to their equivalence to the WLS filters. Next, improved WLS filter designs are obtained, justified by typical properties of the interpolation method that generates the EWD filters. Thus, the digitizing error can be dramatically reduced by adding a one- or two-step delay and slightly increasing the number of the numerator coefficients relative to the denominator. Finally, a choice of suitable implementations of fractional delay filters is available for the various cases of fixed or variable delays.

1. INTRODUCTION

The extended window design (EWD) filters, proposed and analyzed in [1], are IIR digital equivalents of given *analog prototypes*. The nature of their equations which approximate the output signals of analog filters leads to an efficient solution to the problem of implementing fractional delays (FD) by digital means [2] encountered, for example, in sampling rate conversion.

While the efficiency of the EWD approach to FD implementations has been already demonstrated in [2], the main result of the present paper is the proof that the EWD method is near-optimal and equivalent to the traditional weighted least squares (WLS) IIR filter design with frequency sampling [3]. The proof is based on two features of the particular problem considered in this paper. First, it is assumed that the fractional delay is to be applied to the output of a filter that can be designed in analog form. In contrast to the filter design performed directly in the digital domain, the *digitizing of an analog prototype* lends itself to a simple and accurate FD realization. Thus, all the derivations below deal with the problem of approximating the transfer function $H(s)$ of an analog filter by a discrete-time transfer function $H_{EWD}(z)$. Second, since both $H(s)$ and $H_{EWD}(z)$ are analytical rational functions, the properties of such functions dramatically simplify the present problem relatively to the conventional WLS filter design which is meant to approximate an arbitrary frequency response. In fact, the simplest EWD filter design will be shown to yield exactly the same IIR filter as a modified WLS filter design.

Finally, the above equivalence will be used to derive new properties of both classes of digital filters. First, the EWD filters are characterized by the fact that their digitizing error can be reduced by slightly increasing the order of the filter while all the additional poles are placed at z=0. An even more dramatic error reduction is obtained by adding a one- or two-step delay to the designed digital filter. This result leads to improved WLS and iterative WLS filter designs, with or without additional delays. Conversely, the EWD filters are shown to be near-optimal due to the equivalence between the EWD and WLS methods. The importance of these results is twofold: the proposed simplified and enhanced WLS design of conventional digital filters is simpler than the EWD, yet the EWD can be viewed as a better alternative to previous designs of fractional delay filters [4], [5].

The paper is organized as follows. Section 2 briefly presents two forms of the EWD filter equations. The first one is a set of equations derived through linear algebra and numerical methods which demonstrates the efficiency of the EWD filters in implementing fractional delays. The second one is a closed form expression of the EWD transfer function which will be used in Section 3 to prove the equivalence between EWD and WLS filters. Next, Section 3 introduces a modified frequency-domain IIR filter design method, referred to as the matched-pole (MP) frequency sampling design, and gives the proof of the identity of the MP and EWD digital filters. The paper concludes with Section 4.

2. A FEW REPRESENTATIONS OF THE EWD FILTERS

The EWD filters are defined in [1] as digital equivalents of an *analog prototype* with the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{N(s)}{D(s)}, \tag{1}$$

where $N(s)$ and $D(s)$ are known polynomials of orders m_A and n_A, respectively. Without loss of generality, the time is assumed to be normalized to the sampling period of the input signal, and so the *folding frequency* will be $\omega_f = \pi$.

The EWD filters are the result of a *joint time-frequency method* which is based on the interpolation of the input signal $x(t)$ at the sampling times $t=k$, as shown in Fig. 1 below. In addition to targeting the time domain $[k\text{-}m,k]$, the EWD interpolation targets a frequency domain $\omega \in [\omega_0, \omega_M] \subset [0,\pi]$ by expressing the input signal in terms of the trigonometric polynomial

$$x(t) = \sum_{n=0}^{M} (\alpha_n \cos \omega_n t + \beta_n \sin \omega_n t), \quad k-m \le t \le k, \tag{2}$$

where $\omega_0, \ldots, \omega_M$ are distinct *frequency knots* of the interpolation problem. The frequency knots are usually chosen to be equally spaced within the frequency range $[\omega_0, \omega_M]$ that contains most of the spectral energy of $H(j\omega)$. Also, in the basic EWD method, M and m are chosen such that the coefficients α_n and β_n are obtained as the *exact solution* of the algebraic equations

$$\sum_{n=0}^{M} (\alpha_n \cos \omega_n t_k + \beta_n \sin \omega_n t_k) = x(t_k), \tag{3}$$

where $x(t_k)$ are the values of the input signal at the sampling times

t_k=k-m ,..., k. Next, a decision is to be made about an option that was shown in [2] to increase the accuracy of the digitizing procedure. Specifically, an integer delay d may be chosen such that the digital filter output $y_D(k)$, calculated at the current time t=k, actually represents the analog output corresponding to the time instant t=k-d. The EWD recursive generation of the current output $y(k\text{-}d)$ in Fig. 1 is based on the response $y(t)$, $k\text{-}d\text{-}n_A \le t \le k\text{-}d$, of the analog prototype $H(s)$ to the interpolated signal $x(t)$ built from the last $(m+1)$ input samples up to the time t=k. Obviously, any positive integer d produces a digital filter whose implementation implies an output delay d. Moreover, $y(t)$ is determined along the time segment, $k\text{-}d\text{-}n_A \le t \le k\text{-}d$, while the initial conditions are the last n_A output samples, $y(k\text{-}d\text{-}n_A)$, . . . , $y(k\text{-}d\text{-}1)$. Thus, this design, referred to as the *extended window design* (EWD) provides a natural match for the initial conditions of the analog and digital filters and incorporates the interpolation step into the s- to z-domain mapping step. The "current sets" of input and output samples used as *auxiliary conditions* during the computation of the analog filter response are grouped into the vectors $\mathbf{x}_{aux}$ and $\mathbf{y}_{aux}$:

$$\begin{cases} \mathbf{x}_{aux} \triangleq \left[x_k,\, x_{k-1},\, \dots,\, x_{k-m} \right]^T, \\ \mathbf{y}_{aux} \triangleq \left[y_{k-d-1},\, y_{k-d-2},\, \dots,\, y_{k-d-n_A} \right]^T . \end{cases} \tag{4}$$

2.1 The Fractional Delay Form of the EWD Filter Equations

In order to evidence the FD features of the EWD method, this section briefly presents the original design of the EWD filters which used the Chebyshev series representation of signals in terms of vectors of Chebyshev series coefficients [1]. For convenience, the mapping of square integrable functions into the space of square summable vectors defined over the field of real numbers was referred to as the A-transformation. Basically, the A-transform $\mathbf{y}$ of a signal $y(t)$, $t \in [T_1, T_2]$, is the vector $\mathbf{y}=Ay(t)$, whose components are the coefficients of the Chebyshev series expansion of $y(t)$. The inverse A-transform is defined by

$$y(t) = [0.5,\, c_1(\tau),\, c_2(\tau), \dots]\,\mathbf{y}, \qquad \tau = \frac{2\,t-(T_2+T_1)}{T_2-T_1}, \tag{5}$$

where $c_n(\tau)=\cos(n \arccos \tau)$ are the Chebyshev polynomials of the first kind, and τ is a normalized time variable. It is worth noting that, for any given τ and length N_y of $\mathbf{y}$, *only* N_y *multiplications are required*. Indeed, the Clenshaw algorithm (see, e.g., P2 in [2]), which is used to compute (5), does not need the explicit computation of the values $c_n(\tau)$. Assume now that $\mathbf{y}$ is the A-transform of the output signal $y(t)$ of an analog filter, defined in Fig. 1 on the interval $k\text{-}d\text{-}n_A \le t \le k\text{-}d$, corresponding to $-1 \le \tau \le 1$. Then, this vector, restricted to the first N_c coefficients may be used to generate the vector $\mathbf{y}_0$ of the interpolated output,

$$\mathbf{y}_o = \mathrm{P}_o\,\mathbf{y}, \qquad \mathrm{P}_o :\ N_o \times N_c , \tag{6}$$

calculated at any given times $t_o = \{t_i\}_{i=1,2,\dots,No}$ contained in interval $[k\text{-}d\text{-}n_A, k\text{-}d]$. The i^{th} row of the matrix P_o is $[0.5, c_1(\tau_i), c_2(\tau_i), \dots]$ calculated with (5), where $T_1=k\text{-}d\text{-}n_A$, $T_2=k\text{-}d$, τ_i corresponds to t_i, and $-1 \le \tau_i \le 1$.

Now, (2) is used to compute the A-transform $\mathbf{x}$ of $x(t)$ on the subinterval $k\text{-}d\text{-}n_A \le t \le k\text{-}d$ of $k\text{-}m \le t \le k$ in terms of the vector $\mathbf{x}_{aux}$ defined in (4). The result is the following matrix relation

$$\mathbf{x} = \mathrm{C}(m, n_A, d)\,\mathbf{x}_{aux} , \tag{7}$$

which defines the *A-transform interpolation operator* $\mathrm{C}(m,n_A,d)$.

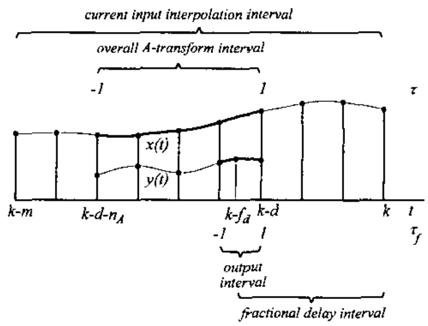

Fig. 1. Interpolation intervals for m=9, n_A=4, and d=3. Definitions of some normalized variables: t = the time normalized to the sampling period; τ = the A-transform time variable of the filter design interval $[k\text{-}d\text{-}n_A, k\text{-}d]$; τ_f = the A-transform variable of the fractional delay interpolation interval $[k\text{-}d\text{-}1, k\text{-}d]$.

As the A-transform reduces the computation of the response of the analog filter $H(s)$ to a linear algebra problem, a first general expression for the A-transform $\mathbf{y}=Ay(t)$ is obtained,

$$\mathbf{y} = -\mathrm{G}\,\mathbf{y}_{aux} + \mathrm{F}\,\mathbf{x}_{aux}. \tag{8}$$

The parameters G and F in (8) are calculated once and for all for any given transfer function $H(s)$. Then, the inverse A-transform (5) is to be applied to both sides of (8) in order to yield $y(t)$ at any desired time instant $k\text{-}d\text{-}n_A \le t \le k\text{-}d$. In particular, the choice t=k-d provides the conventional digital filter difference equation

$$y_{k-d} = -\mathbf{g}^T\mathbf{y}_{aux} + \mathbf{f}^T\mathbf{x}_{aux}. \tag{9}$$

Likewise, any fixed fractional delay like f_d in Fig. 1 yields a pair $\{\mathbf{g},\mathbf{f}\}$ of vectors which can be pre-computed and saved. A different problem arises when f_d is variable and must be determined in real time, during each sampling interval. An efficient solution, which avoids the recalculation of $\mathbf{g}$ and $\mathbf{f}$ from G and F, is available based on the *time-window contraction matrix* S (see, e.g., [2]) which relates the A-transforms $\mathbf{y}$ and $\mathbf{y}_C$, respectively defined on the intervals $-1 \le \tau \le 1$ and $-1 \le \tau_f \le 1$ in Fig. 1: $\mathbf{y}_C = \mathrm{S}\,\mathbf{y}$. Now, the solution (8), together with the matrices $\mathrm{C}_y = \mathrm{S}\,\mathrm{G}$ and $\mathrm{C}_x = \mathrm{S}\,\mathrm{F}$, is calculated once and for all, under the assumption that the *output interval* is placed as in Fig. 1, and only the fractional part of the delay is variable. This means that d will also be fixed and equal to the minimum possible value of the integer part of the delay, whereas any additional integer corresponding to the current delay will be introduced during the digital implementation of the filter.

Property P1: Efficiency of the FD Computation. It can be shown [2] that, due to the small length of the *output interval* relatively to the length of the *overall A-transform interval*, only very few terms of the matrices C_y and C_x are needed. Also, usually, the matrix C_x can be ignored altogether if d=0, as the latest input samples have little effect on the current output interval $[k\text{-}1,k]$. Thus, the current A-transform $\mathbf{y}_C$ requires only a few operations and is given by

$$\mathbf{y}_C = -\mathrm{C}_y\,\mathbf{y}_{aux} + \mathrm{C}_x\,\mathbf{x}_{aux} , \tag{10}$$

where the vector $\mathbf{y}_{aux}$ is updated with the original equation (9). Finally, for each new value t_o=k-f_d of the interpolation time, the inverse A-transform (5) applied to $\mathbf{y}_C$ yields the interpolated output value y_0 with one multiplication for each component of $\mathbf{y}_C$.

Property P2: Digitizing Error Reduction when $d \geq 1$, and $m > n_A + d$. The well-known fact that the interpolation at equally spaced points is more accurate toward the center of the interpolation interval [6] can be used to advantage when the fractional delay interval f_d has an integer part $d \geq 1$. Thus, according to Fig. 1, the last d sampling periods prior to the current input sample can be left out of the overall A-transform interval used in the computation of $\mathbf{y}$. This means that the A-transforms $\mathbf{x}$ and $\mathbf{y}$ will be calculated only for the *overall A-transform interval* $k-d-n_A \leq t \leq k-d$ in Fig. 1. Moreover, a slight increase of m beyond n_A+d improves the accuracy of the approximation of $y(t)$ as most of the input signal energy used to calculate its A-transform $\mathbf{y}$ will be provided by the accurate central segment of the approximation of $x(t)$ on the interval $[k-d-n_A, k-d]$.

2.2 The EWD Transfer Function

The transfer function $H_{EWD}(z)$ below, corresponding to the EWD difference equation (9), was derived in [7] for any integer d that satisfies the condition $0 \leq d \leq m-n_A$. This expression will be used in Section 3 to prove the equivalence between the EWD and WLS filters.

Let $x(t)$ be the trigonometric polynomial (2) whose coefficients α_n and β_n are obtained by solving (3) for the particular right-hand side values $x(t_k)$ all equal to zero with the exception of the last sample which is one. Also let the origin of the time axis t be placed at $t=k-d-n_A$. Then the Laplace transform $X(s)$ of $x(t)$, and the z-transform $X_D(z)$ of the sampled sequence $x_D(k)$ are given by

$$X(s) = \frac{R(s,m,L)}{\prod_{n=0}^{M}(s^2+\omega_n^2)}, \quad X_D(z) = \frac{z^{m+1-L}}{\prod_{n=0}^{M}(z^2-2z\cos\omega_n+1)}, \tag{11}$$

where $L=n_A+d$, and $R(s,m,L)$ is a polynomial in s. In the following, $Z\{Y(s)\} \triangleq Z\{[\mathcal{L}^{-1}[Y(s)]]|_{t=k}\}$ will denote the z-transform of the samples of $y(t)$, and the denominators in (11) will be written as

$$Q(s) = \prod_{n=0}^{M}(s^2+\omega_n^2), \quad Q_D(z) = \prod_{n=0}^{M}(z^2-2z\cos\omega_n+1). \tag{12}$$

This yields the design equations

$$\begin{aligned} Z\{X(s)H(s)\} &= Z\left\{\frac{R(s,m,L)\,H(s)}{Q(s)}\right\} \\ &\triangleq Z\left\{\frac{b_0 s^{n_A+m-1}+\ldots+b_{n_A+m-1}}{Q(s)\,(s^{n_A}+a_1 s^{n_A-1}+\ldots+a_{n_A})}\right\} \\ &\triangleq \frac{z(\beta_0 z^{n_A+m}+\ldots+\beta_{n_A+m})}{Q_D(z)\,(z^{n_A}+g_1 z^{n_A-1}+\ldots+g_{n_A})}, \end{aligned} \tag{13}$$

and the final causal transfer function

$$H_{EWD}(z) = \frac{Q_D(z)}{z^{m+1-n_A}}\left[\frac{z(\beta_0 z^{n_A+m}+\ldots+\beta_{n_A+m})}{Q_D(z)\,(z^{n_A}+g_1 z^{n_A-1}+\ldots+g_{n_A})} + \frac{z(c_1 z^{n_A-1}+\ldots+c_{n_A})}{z^{n_A}+g_1 z^{n_A-1}+\ldots+g_{n_A}}\right]. \tag{14}$$

The following recursive equations determine the coefficients c_k:

$$Q_D(z) \triangleq z^{m+1}+d_1 z^m+\ldots+d_{m+1}, \quad c_1 = -\beta_0,$$
$$\left\{c_k = -\beta_{k-1} - \sum_{n=1}^{k-1} c_{k-n} d_n, \quad k=2,\ldots,n_A\right\}. \tag{15}$$

With (15), the n_A leading terms of the numerator of $H_{EWD}(z)$ are canceled to satisfy the standard *condition of initial rest* defined in terms of the auxiliary conditions (4). The final $H_{EWD}(z)$ is precisely the transfer function corresponding to (9). The algebraic analysis of (14) yields Property *P3* below.

Property P3: The real-time implementation of the EWD transfer function is of order $n_D = m \geq n_A+d$, with the n_A poles z_n related to the poles s_n of the analog prototype by the expression $z_n = e^{s_n}$, while the remaining poles, if any, are placed at $z=0$.

3. EQUIVALENCE BETWEEN EWD AND WLS FILTERS

Consistently with the approach taken in this paper, the frequency domain IIR filter design problem is restricted to the problem of approximating the transfer function (1) of an analog filter of order n_A by a discrete-time transfer function

$$H_D(z) = \frac{p_0 z^m + p_1 z^{m-1}+\ldots+p_m}{z^{m-n_A}(z^{n_A}+g_1 z^{n_A-1}+\ldots+g_{n_A})} = \frac{N_D(z)}{D_D(z)}, \tag{16}$$

where $m \geq n_A$. Since both $H(s)$ and $H_D(z)$ are analytical rational functions, the properties of analytic functions imply some rather powerful constraints on their behavior within their respective regions of convergence. Moreover, these constraints also restrict the behavior of $H(j\omega)$ and $H_D(e^{j\omega})$ leading, for example, to the relationships between their real and imaginary parts, as well as their magnitudes and phase angles.

3.1 The Matched-Pole Frequency Sampling Design

One more consequence of the constraints on the frequency responses $H(j\omega)$ and $H_D(e^{j\omega})$ relates the poles of $H(s)$ and $H_D(z)$ through the relationship $z_n = e^{s_n}$, mentioned in Property *P3* above. This is equivalent to the time domain condition that the modes z_n^k of the digital filter are exactly the same as the sampled sequences of the modes $e^{s_n t}$ of the analog prototype. Moreover, in theory, the analyticity properties imply that $H(j\omega)$ and $H_D(e^{j\omega})$ can be completely defined by their values on finite intervals of the real frequency ω. Yet, the numerical design below is based on the pole-matching condition, and aims at matching the two frequency responses within a frequency range $[\omega_0, \omega_M]$ that contains most of the spectral energy of $H(j\omega)$. Based on the above considerations, a modified version of the conventional IIR frequency sampling design [3] is now proposed. The set of algebraic equations

$$N_D(e^{j\omega_n}) = e^{-jd\omega_n} D_D(e^{j\omega_n})\,H(j\omega_n), \quad n=0,\ldots,M, \tag{17}$$

is solved for the $(m+1)$ coefficients p_n defined in (16), while $D_D(z)$ is chosen with the same coefficients g_k, $k=1, \ldots, n_A$, as those found in the vector $\mathbf{g}$ in (9) and the denominator of $H_{EWD}(z)$ in (14). Here, M and the *frequency knots* $\omega_0, \ldots, \omega_M$ are chosen in the same way as in (2), that is such that (17) has a unique solution. The optional delay $d>0$ makes the output $y_D(k)$, calculated at the current time $t=k$, actually represent the analog output at the time instant $t=k-d$.

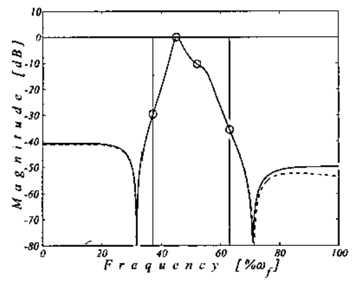

Fig. 2 Normalized responses $H(j\omega)$ (solid line), and $H_D(e^{j\omega})$ (dashed line) for $H(s) = \frac{(s^2+1)(s^2+5)}{(s^2+0.1s+2)(s^2+0.3s+2)(s^2+0.2s+3)}$.

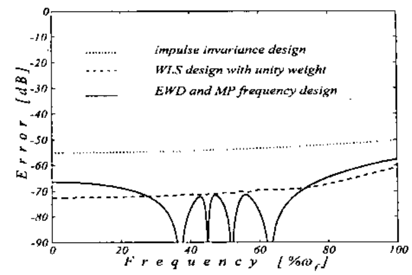

Fig. 3. The digitizing error $E_D(\omega)$ of four digital filters equivalent to the transfer function $H(s)$ shown in Fig.2 (n_A=6, m=7, and d=0).

Since the IIR filter design is now reduced to an FIR frequency-sampling problem [3], this method will be referred to in the following as the *matched-pole (MP) frequency sampling design.*

3.2 Identity of the EWD and MP Frequency Sampling Design

To prove that the EWD equation (9), as well as (13)-(15), yields *exactly* the same transfer function (16) as the MP frequency sampling design it suffices to show that they produce the same numerator $N_D(z)$. First of all, the EWD computes the numerator coefficients such that the digital filter response coincides with the (possibly delayed) sampled response of the analog filter to the interpolated input. Moreover, as (3) has a unique solution, inputs like $x_n(t) = e^{j\omega_n t}$, $\omega_n = \omega_0, \ldots, \omega_M$, are exactly interpolated (with *an optional d-step delay* $d \geq 0$). Therefore, the exact response of $H(s)$ is $y_n(t) = H(j\omega_n)e^{j\omega_n(t-d)}$, while the EWD output is given by the relation $y_n(k) = H(j\omega_n)e^{j\omega_n(k-d)} = H_{EWD}(e^{j\omega_n})e^{j\omega_n k}$, and so the EWD filter satisfies the expression

$$H_{EWD}(e^{j\omega_n}) = e^{-jd\omega_n}H(j\omega_n), \quad n = 0, \ldots, M. \tag{18}$$

Finally, since the MP design determines the numerator coefficients p_n as the exact solution of (17), the comparison of (17) and (18) leads to the conclusion that $H_{EWD}(z) = H_D(z)$, which proves the identity of the MP frequency sampling filters and the EWD filters for integer values of the delay $d \geq 0$.

Now, it is worth noting the close relationship between the EWD and MP filter design methods and the traditional WLS design [3]. The latter minimizes the expression

$$\min_{p_\ell, g_\ell} \sum_{n=0}^{M} w(\omega_n) \left| e^{-jd\omega_n}H(j\omega_n) - H_D(e^{j\omega_n}) \right|^2, \tag{19}$$

where p_ℓ and g_ℓ are the unknown coefficients of the numerator and denominator of $H_D(z)$, ω_n are the frequency knots within $[0,\pi]$, and $w(\omega)$ is a user defined weighting function. As the IIR filter design considered in this paper deals with the problem of approximating the transfer function of an analog filter by a discrete-time transfer function, the analyticity properties discussed above state that the denominator of $H_D(z)$ is uniquely computed through the mapping defined in Property *P3*. Indeed, extensive tests done on the EWD, MP, and WLS designs support this assertion, in the sense that all the denominators are practically the same, even when a large number M is chosen in (19). In this last case, the WLS design may lead to slightly different numerators, while the values of the digitizing error $E_D(\omega) = 20\log_{10}(|e^{-jd\omega}H(j\omega) - H_D(e^{j\omega})|)$ remain about the same as those obtained with the EWD and MP methods. This is illustrated in Figs. 2 and 3. Actually, depending on the weighting function $w(\omega)$, the shape of the error $E_D(\omega)$ provided by the WLS design may vary all the way from the dashed line in Fig. 3, where $w(\omega) = 1$, to the shape of the EWD error, shown by the solid line. Also, Fig. 2 evidences the four frequency knots ω_n (chosen within the narrow frequency range [37,63] % ω_f) which define $H_D(z)$ under the pole-matching condition and consistently with the analysis based on the analyticity properties. Moreover, if d=1, then the EWD, MP, and WLS digitizing errors decrease by about 20 dB.

4. CONCLUSIONS

The proof that the EWD and WLS filters are equivalent implies that the EWD filters are near-optimal, and so they are not only efficient tools for FD applications, but also very accurate in reproducing the characteristics of the original filters (designed in analog form) whose delays are to be controlled. Next, according to Property *P2*, the digitizing errors can be reduced by adding a small delay, or by increasing the filter order while the additional poles are placed at z=0. Usually, such delays are negligible with respect to the inherent group delay of the original filters. At the same time, it ensues that the frequency sampling methods can be improved in exactly the same way, due to their equivalence to the EWD method. Moreover, a choice of suitable FD filter implementations is available for various applications. Thus, fixed fractional delays, can be realized with either the EWD or frequency sampling methods improved through the pole-matching condition. Finally, variable delays required, for example, in sampling rate conversion can be efficiently realized with the EWD equations (9) and (10).

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